Hardness of Approximate Coloring

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PhD Defence

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Graph Coloring

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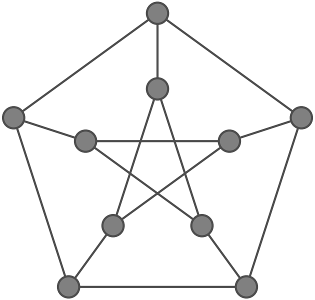

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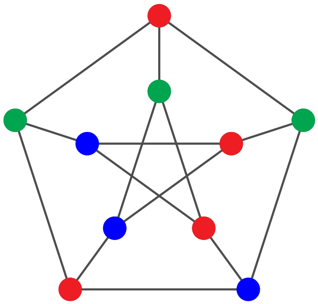

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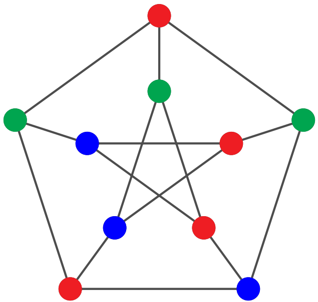

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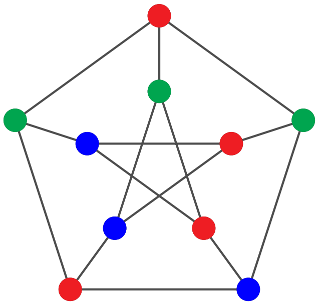

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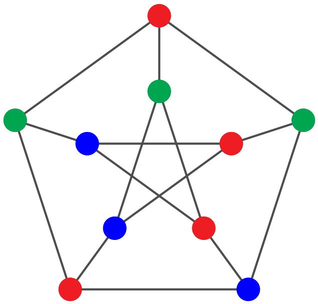

Goal : Color a graph using **few colors**, **efficiently**.

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Dream Goal

$$\exists \delta > 0 :$$

hard to efficiently color 3-colorable graphs with n^{δ} colors

assuming $\mathsf{P} \neq \mathsf{NP}$.

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Theorem (Dream Goal)

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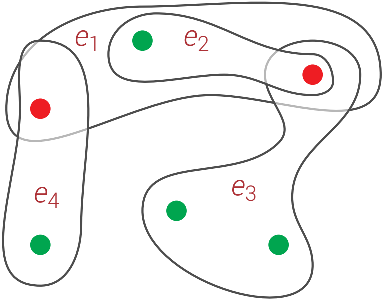
Can we achieve dream goal for hypergraphs?



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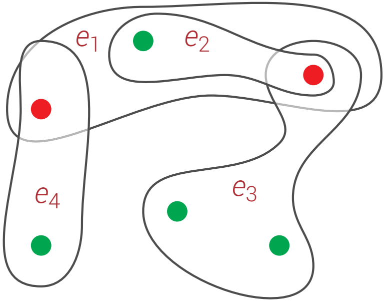

Theorem (Dream Goal)

Given a 2-colorable k-uniform hypergraph,

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Result - II : Hardness of Hypergraph Coloring

For 2-Colorable Hypergraphs

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Definition (CSP for a Predicate $P \subseteq \{0,1\}^k$)

Given a k-uniform hypergraph $G = (V, E)$, find assignment to vertices $f : V \to \{0,1\}$ such that $\forall e \in E, f|_e \in P$.

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Definition (CSP for a Predicate $P \subseteq \{0,1\}^k$ with literals)

Given a k-uniform hypergraph $G = (V, E)$ and literal function $L : E \to \{0,1\}^k$, find assignment to vertices $f : V \to \{0,1\}$ such that $\forall e \in E, f|_e \oplus L(e) \in P$.

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G is 2-colorable iff

the CSP instance with $P =$ NOT ALL EQUAL is satisfiable.

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Definition (Covering Number)

Smallest number c such that there exists c assignments that together satisfies all edges.

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For NOT ALL EQUAL, 2-coloring $\equiv$ covering number $= 1$.

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For odd predicates, covering number ≤ 2.

Any assignment and its complement covers all the edges.

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- For the 4-LIN predicate, given a 2-coverable instance, it is hard to find an independent set of relative size $1/\log n$.

Techniques

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Check $\{\text{Proof}_i\}$
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$\mathrm{OPT(LC)} = 1$	$\exists$ a proof over q with $\Pr[\mathrm{accept}] = 1$.	$\mathcal{G}$ is q-colorable.

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$\text{OPT}(LC) = 1$	$\exists$ a proof over q with $\Pr[\text{accept}] = 1$.	$\mathcal{G}$ is q-colorable.
$\text{OPT}(LC) \leq \delta$	$\forall$ proofs over Q, $\Pr[\text{accept}] < 1 - \delta^{c_Q}$.	$\mathcal{G}$ has no independent set of size $\lvert\mathcal{V}\rvert/Q$.

Long Code: From Label Cover to PCP Verifier

Long Code [Bellare Goldreich Sudan '95]

The long code of $a \in [L]$ is $A : \{0,1\}^L \to \{0,1\}$ such that $A(f) = f_a$.

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- Soundness : If test passes with high probability then A can be explained by a short list of long codes.

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- PCP proof consists of long code encoding of labels of vertices in Label Cover instance.
- Due to parallel repetition $L = \text{poly}(1/\delta)$ and $|V| = n^{\log(1/\delta)}$.

Proof size is $2^{\text{poly}(1/\delta)}$. Cannot go beyond $\delta = O(1/\text{poly}\log n)$.

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Test for Low Degree Long Codes

Choose $f_1, \cdots, f_k$ of degree $\leq d$. Accept if $\{A(f_i)\}$ are Not-All-Equal.

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(Above the second arrow: ~~Long Code~~; below it: Low Degree Long Code)

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Test on Not-All-Equal predicate can be obtained either by

- increasing the number of queries to 8, or
- increasing the alphabet to $\mathbb{F}_2 \times \mathbb{F}_2$.

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- Combining [Khot Saket '14] label cover with our tests yields:
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  - Can't color 4-colorable 4-uniform hypergraphs with $2^{\Omega(\log^\delta n)}$ colors.

3-colorable 3-uniform Hypergraphs

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Long Code Test of [Khot '02]

3-colorable 3-uniform Hypergraphs

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- Choose uniformly random $f, g : \mathbb{F}_3^\ell \to \mathbb{F}_3$ and $\eta : \mathbb{F}_3^\ell \to \{1, 2\}$.

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in Soundness Analysis of Low Degree Long Code Test

[Dinur Guruswami '13]

If $\alpha : \mathbb{F}_2^{\ell} \to \mathbb{F}_2$ such that $\mathrm{dist}(\alpha, \mathrm{P}_{\ell-d-1}) > 2^d$ then

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Techniques

Result - I : Hardness of Graph Coloring

Graph Product

Graph Product

Definition

Given a graph $G = (V, E)$, the n-wise product graph $G^n := (V^n, E^n)$ where

$$E^n := \left\{ ((u_1, \ , u_n), (v_1, \ , v_n)) : \bigwedge_{i \in [n]} (u_i, v_i) \in E \right\}.$$

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Alon, Dinur, Friedgut, Sudakov 2004, Dinur, Mossel, Regev '07 & Dinur Shinkar '10

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A subset of $\{0, 1, 2\}^n$ which fixes some coordinate x_i to $a \in \{0, 1, 2\}$.

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4. **A is explained by a set of dictators of size $1/\text{poly}(\delta)$.**

Our Result

There exists a subgraph $G' = (V', E')$ of $V(K_3^n)$ of size $3^{\text{poly}(\log n)}$ such that

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Let A be an independent set in G' of relative size δ. Then,

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For any i, $f(i) - g(i) = h(i)^2 + 1 \in \{1,2\}$.

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That is A is close to a function with only degree 1 Fourier terms.

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Theorem (FKN '04 & ADFS '04)

If $A : \{0, 1, 2\}^N \rightarrow \{0, 1\}$ is close to a function having only degree 1 Fourier terms then A is close to a dictator.

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If $|\alpha| > 3^d$ and $h \in_U P_{d/2}$ then

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Publications : Conference

- Venkat Guruswami, Prahladh Harsha, Johan Håstad, Srikanth Srinivasan, & Girish Varma. *Super-polylogarithmic hypergraph coloring hardness via low-degree long codes*. *Symp. on Theory of Computing (STOC)*, 2014.
- Irit Dinur, Prahladh Harsha, Srikanth Srinivasan, & Girish Varma. *Derandomized graph product results using the low degree long code*. *Symp. on Theoretical Aspects of Computer Science (STACS)*, 2015.
- Amey Bhangale, Prahladh Harsha, & Girish Varma. *A characterization of hard-to-cover CSPs*. *Computational Complexity Conference (CCC)*, 2015.

Publications : Manuscript

- Girish Varma. *Reducing uniformity in Khot-Saket hypergraph coloring hardness reductions*. CoRR.

That’s all Folks!

Thank You